Globally Optimal FIR Filters with Applications in Source and Channel Coding[1]

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Abstract —

In this paper, we derive a novel formulation to solve an important FIR filter optimization problem. The problem has received considerable attention in the past because it appears in a wide variety of disciplines. The newly proposed method finds the globally optimal solution to the problem and provides several other advantages over previous optimization techniques.

I. Introduction

Consider the following optimization problem

$$\max_{H(e^{j\omega})} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 W(e^{j\omega}) \frac{d\omega}{2\pi} \tag{1}$$

subject to

$$\frac{1}{M}\sum_{k=0}^{M-1} |H(e^{j(\omega-2\pi k/M)})|^2 = |H(e^{j\omega})|^2|_{\downarrow M} = 1 \tag{2}$$

where $H(e^{j\omega})$ is a real FIR filter of order N. The constraint (2) means that the magnitude squared response $|H(e^{j\omega})|^2$ is Nyquist(M). Depending on the choice of the frequency weight function $W(e^{j\omega})$, the problem described above appears in the context of both channel and source coding (See [1] for details).

II. The new approach

From (1) and (2), we can immediately observe that any solution is only a function of $|H(e^{j\omega})|^2$. By denoting the product filter $H(z)H(z^{-1})$ by $F(z)$, the objective function (1) can be rewritten as $w(0) + 2\sum_{n=1}^{N} f(n)w(n)$ and the constraint (2) becomes

$$f(Mn) = \delta(n) \tag{3}$$

$$F(e^{j\omega}) \geq 0 \quad \forall\, \omega \tag{4}$$

where $w(i)$ denotes the inverse discrete time Fourier transform of the $W(e^{j\omega})$. The major difficulty with the product filter approach is to satisfy the positivity constraint (4). Since $F(z)$ is a two sided symmetric sequence, it can be written as $D(z) + D(z^{-1})$ where $D(z)$ is a causal function that completely characterizes $F(z)$. The key idea is then to exploit the *rationality* and *causality* of the function $D(z)$ to re-express the filtering problem in terms of algebraic conditions rather than analytical ones, i.e., in terms of a state space realization (A_d, B_d, C_d, D_d) of the function $D(z)$ and a positive definite matrix P_d which has the role of insuring the positivity of the resulting filter $F(e^{j\omega})$. It can be shown [1] that the original optimization problem reduces to the following final form:

$$\max_{C_d, P_d} \quad C_d\, \mathbf{R}^T - Tr(\mathbf{W}P_d) \tag{5}$$

where $\mathbf{R}^T = [w(N)\ldots w(1)]^T$ and $\mathbf{W} = \alpha\mathbf{I}$ is a fixed diagonal positive semi definite weight matrix such that

$$\begin{aligned} \mathcal{M}_d &= \begin{bmatrix} P_d - A_d^T P_d A_d & C_d^T - A_d^T P_d B_d \\ C_d - B_d^T P_d A_d & D_d + D_d^T - B_d^T P_d B_d \end{bmatrix} \succeq 0 \\ \mathbf{Q}\, C_d^T &= \mathbf{0} \end{aligned} \tag{6}$$

where $\mathbf{0}$ is the $(N-1)\times 1$ zero vector. The above formulation is therefore a maximization problem in the variable vector C_d and a minimization problem in the matrix P_d. The state space variables are given by :

$$\begin{aligned} A_d &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ 0 & \mathbf{0}^T \end{bmatrix}, \quad B_d = [\,0\,0\ldots\,1\,]^T, \\ C_d &= [\,f(N)\,\ldots\,f(1)\,], \quad D_d = \frac{1}{2} \end{aligned} \tag{7}$$

III. Advantages of the new method

1. Unlike previous methods, the positivity constraint is satisfied over all ω. For a fixed weight function $W(e^{j\omega})$, the resulting filter $F_{opt}(z)$ is guaranteed to be the *global* optimum due to the convexity of the new formulation.
2. Equations (5) and (6) describe a multi-objective *semi-definite* program which can be solved quite efficiently using recently developed interior point methods [2]. The algorithm is numerically robust and numerical sub-optimality can be controlled by using a primal dual interior point algorithm [2].
3. The new formulation is extremely general in the sense that it works for any given M, any chosen filter length N and any weight function (not necessarily positive) $W(e^{j\omega})$.
4. A spectral factorization step, required to obtain $H(z)$ from $F_{opt}(z)$, is avoided. Spectral factorization is, in general, a computationally expensive operation that can be numerically unstable. The *minimum phase* spectral factor can be expressed in terms of the matrices (A_d, B_d, D_d), the optimal vector $C_{d_{opt}}$ and, $P_{d_{min}}$, where $P_{d_{min}}$ is *the minimum element* in the convex cone of positive definite matrices satisfying (6). The exact expressions can be found in [1].
5. Additional linear constraints such as wavelets regularity constraints can be easily added.

References

[1] J. Tuqan and P. P. Vaidyanathan, "The role of the discrete-time Kalman-Yakubovich-Popov (KYP) lemma in designing statistically optimum FIR orthonormal filter banks," *Invited paper, Proc. IEEE ISCAS* , Monterey, CA, June 1998.

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